❑ A NEW VIEW OF PASCAL'S TRIANGLE AND THE BINOMIAL THEOREM

Notice that to write a certain row of Pascal's Triangle, we need the row above it, which of course requires the row above that one, etc. This can be cumbersome; we need a way of finding any row of Pascal's Triangle directly – without having to figure out all the rows that precede it.

[Link to chapter on counting]

Consider the sequence of combinations, from "6 choose 0" to "6 choose 6":

$$\binom{6}{0},\ \binom{6}{1},\ \binom{6}{2},\ \binom{6}{3},\ \binom{6}{4},\ \binom{6}{5},\ \binom{6}{6}$$

Evaluating each one of these (in other words, you verify!) gives the sequence of numbers:

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

which is none other than the 6th row of Pascal's Triangle (if we count the top row as the 0th row). So, to expand $(a + b)^6$, rather than using the appropriate row of Pascal's Triangle, we can use the sequence of combinations instead.

$$(a+b)^6$$

$$= \binom{\mathbf{6}}{\mathbf{0}}a^6b^0 + \binom{\mathbf{6}}{\mathbf{1}}a^5b^1 + \binom{\mathbf{6}}{\mathbf{2}}a^4b^2 + \binom{\mathbf{6}}{\mathbf{3}}a^3b^3 + \binom{\mathbf{6}}{\mathbf{4}}a^2b^4 + \binom{\mathbf{6}}{\mathbf{5}}a^1b^5 + \binom{\mathbf{6}}{\mathbf{6}}a^0b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Similarly, to start the expansion of $(a + b)^{100}$, we would write

$$\binom{100}{0}a^{100}b^0 + \binom{100}{1}a^{99}b^1 + \binom{100}{2}a^{98}b^2 + \ldots$$

$$= \quad \boldsymbol{a^{100} + 100a^{98}b + 4{,}950a^{98}b^2 + \ldots}$$

The interesting thing here is that we've begun the expansion of $(a + b)^{100}$ without generating all the preceding rows of Pascal's Triangle.

We now tie everything in this chapter into a single formula which will calculate $(a + b)^n$ without any reference to Pascal's Triangle.

The Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Homework

1. Use the Binomial Theorem to prove: $(x + y)^2 = x^2 + 2xy + y^2$.

2. Find the first four terms of the expansion of $(a + b)^{20}$.

3. Use the Binomial Theorem to expand $(x + y)^5$.

Review Problems

4. Consider the expansion of $(x + y)^{99}$. How many terms are there in the expansion? In the term containing y^{33}, what is the exponent on the x?

5. There are 250 terms in the expansion of $(a + b)^n$. What is n?

6. Find the 11th row of Pascal's Triangle (the one which starts 1 11 . . .)

7. Find the last two terms of $(a + b)^{66}$.

8. Use Pascal's Triangle to expand $(a + b)^8$.

9. Use the Binomial Theorem to prove: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

10. Find the first five terms of the expansion of $(a + b)^{30}$.

11. Use the Binomial Theorem to expand $(a + b)^7$.

12. True/False:

    a. There's a power of $a + b$ that's equal to 1.
    b. The expansion of $(x + y)^{123}$ contains 123 terms.
    c. The first term of the expansion of $(a + b)^{55}$ is $55a^{55}$.
    d. In every term of the expansion of $(p + q)^m$, the sum of the exponents is m.
    e. The expansion of $(a + b)^{300}$ begins with $a^{300} + 300a^{299}b$.
    f. Every natural number (1, 2, 3, . . .) occurs in at least one place in Pascal's Triangle.

g. If there are 1000 terms in the expansion of $(a + b)^n$, then $n = 1001$.

h. Given any element in Pascal's Triangle, there are numbers n and k such that $\binom{n}{k}$ is that element.

i. The expansion of $(a + b)^{50}$ begins with $a^{50} + 50a^{49}b + 1225a^{48}b^2$.

❑ To ∞ and Beyond

A. Use the Binomial Theorem to prove that

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

B. Recall how each row of Pascal's Triangle is generated: Each element is the sum of the two numbers above it. Prove this fact using combinations:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Solutions

1. They're boring, but do them!

2. a. 24 b. a^{23}; b^{23} c. 8

3. $n = 101$

4. 1 10 45 120 210 252 210 120 45 10 1

5. $a^{100} + 100a^{99}b + \cdots + 100ab^{99} + b^{100}$

6. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6$
$+120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$

7.
$$\begin{aligned}(x+y)^2 &= \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 \\ &= 1x^2 \cdot 1 + 2xy + 1 \cdot 1y^2 \\ &= x^2 + 2xy + y^2\end{aligned}$$

8. $a^{20} + 20a^{19}b + 190a^{18}b^2 + 1140a^{17}b^3 + \cdots$

9.
$$\begin{aligned}(x+y)^5 &= \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}x^0y^5 \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\end{aligned}$$

10. Hint: Let $a = 1$ and $b = 1$ in the Binomial Theorem, and then simplify each side of the equation.

11. Hint: Start with the sum of the combinations on the right. Expand each one into its definition using factorials. Find an LCD and add the fractions together. Simplify the numerator and you should end up with "*n* choose *k*," the left side of the equation.

12. 100; 66 **13**. 249

14. 1 11 55 165 330 462 462 330 165 55 11 1

15. . . . + $66ab^{65} + b^{66}$

16. $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

17. $$(a+b)^3 = \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

18. $a^{30} + 30a^{29}b + 435a^{28}b^2 + 4060a^{27}b^3 + 27{,}405a^{26}b^4$

19. $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

20. a. T b. F c. F d. T e. T f. T g. F h. T i. T